PRINCIPLES OF ANALYSIS
LECTURE 11 - LIM SUP AND LIM INF

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1. lim sup and lim inf

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Define

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n \mid n \geq N\}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n \mid n \geq N\}.$$

The sequence $\{\sup\{s_n \mid n \geq N\}_{N=1}^{\infty}$ is decreasing and the sequence $\{\inf\{s_n \mid n \geq N\}_{N=1}^{\infty}$ is increasing, so they both converge, or diverge to $\pm\infty$.

Proposition 1. *Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Then $\liminf s_n \leq \limsup s_n$.*

Lemma 1. *Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be sequences such that $s_n \leq t_n$ for every $n \in \mathbb{N}$. If they both converge, we have $\lim s_n \leq \lim t_n$.*

proof of Lemma. Let $s = \lim s_n$ and $t = \lim t_n$; suppose bwoc that $t < s$. Set $\epsilon = \frac{t-s}{2}$; then there exists $N_1 \in \mathbb{Z}^+$ such that $n \geq N_1$ implies $|s_n - s| < \epsilon/2$, and there exists $N_2 \in \mathbb{Z}^+$ such that $n \geq N_2$ implies $|t_n - s| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$; then by and application of the triangle inequality, $t_n < s_n$, a contradiction. □

proof of Proposition. For every $N \in \mathbb{Z}^+$, we have $\inf\{s_n \mid n \geq N\} \leq \sup\{s_n \mid n \geq N\}$. Thus the result is immediate from the lemma. □

Date: September 15, 2003.

Proposition 2. *Let* $\{s_n\}_{n=1}^\infty$ *be a sequence of real numbers. Then* $\{s_n\}_{n=1}^\infty$ *converges if and only if* $\liminf s_n = \limsup s_n$*, in which case* $\liminf s_n = \lim s_n = \limsup s_n$*.*

Lemma 2. *Let* $x, y \in \mathbb{R}$*. If* $x \leq y + \epsilon$ *for every* $\epsilon > 0$*, then* $x \leq y$*.*

Proof. Suppose that $x > y$, and let $\epsilon = \frac{x-y}{2}$. Then $y+\epsilon = x-\epsilon$, so $x > y+\epsilon$. □

Proof.

($\Rightarrow$) Suppose that $\{s_n\}_{n=1}^\infty$ converges to a real number s. Let $\epsilon > 0$. We wish to show that $\limsup s_n \leq s + \epsilon$ for every $\epsilon > 0$, whence $\limsup s_n \leq s$.

Since $s_n \to s$, there exists $N \in \mathbb{Z}^+$ such that $|s_n - s| < \epsilon$ for $n \geq N$. Then $\sup\{s_n \mid n \geq N\} < s + \epsilon$. Since $\{\sup\{s_n \mid n \geq N\}\}_{N=1}^\infty$ is a decreasing sequence, we have $\limsup s_n < s + \epsilon$. Therefore $\limsup s_n \leq s$.

Similarly, $s \leq \liminf s_n$, so

$$s \leq \liminf s_n \leq \limsup s_n \leq s,$$

so

$$\liminf s_n = s = \limsup s_n.$$

($\Leftarrow$) Now suppose that $\liminf s_n = \limsup s_n$, and label this common value s. We want to show that $\lim s_n = s$.

Let $\epsilon > 0$. Since $s = \limsup s_n$, there exists $N_1 \in \mathbb{Z}^+$ such that

$$|\sup\{s_n \mid n \geq N_1\} - s| < \epsilon.$$

In particular, $\sup\{s_n \mid n \geq N_1\} < s + \epsilon$, so $s_n < s + \epsilon$ for $n \geq N_1$. Similarly, since $s = \liminf s_n$, there exists $N_2 \in \mathbb{Z}^+$ such that $s_n > s - \epsilon$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we have $s - \epsilon < s_n < s + \epsilon$, that is, $|s_n - s| < \epsilon$. Thus $s_n \to s$. □

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